

Exact solutions of $(n + 1)$ -dimensional Yang-Mills equations in curved space-time

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Abstract

In the context of a semiclassical approach where vectorial gauge fields can be considered as classical fields, we obtain exact static solutions of the $SU(N)$ Yang-Mills equations in a $(n + 1)$ dimensional curved space-time, for the cases $n = 1, 2, 3$. As an application of the results obtained for the case $n = 3$, we consider the solutions for the anti-de Sitter and Schwarzschild metrics. We show that these solutions have a confining behavior and can be considered as a first step in the study of the corrections of the spectra of quarkonia in a curved background. Since the solutions that we find in this work are valid also for the group $U(1)$, the case $n = 2$ is a description of the $(2 + 1)$ electrodynamics in presence of a point charge. For this case, the solution has a confining behavior and can be considered as an application of the planar electrodynamics in a curved space-time. Finally we find that the solution for the case $n = 1$ is invariant under a parity transformation and has the form of a linear confining solution.

Keywords: Yang-Mills equations, exact static solutions, semiclassical approach, curved space-time.

1. Introduction

The quark confinement problem [1, 2, 3, 4, 5] cannot be solved by the perturbation theory of quantum chromodynamics (QCD) so long as the confinement is a nonperturbative phenomenon [6]. In essence the question is about how to confine the colour charges at distances of order of characteristic hadronic sizes [7]. The semiclassical approach permits

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to acquire non-perturbative information about QCD starting from the solutions of classical partial differential equations of $SU(3)$ Yang-Mills theory [8, 9]. The searching of solutions of classical Yang-Mills equations in presence of static external sources has been motivated by understanding the role of non-abelian gauge fields of quark confinement problem. One of the first works about this subject showed that if an external source is distributed over a thin spherical shell the Coulomb solution is unstable in a specific regime [7].

A wide range of solutions of $(3 + 1)$ -dimensional Yang-Mills equations in presence of localized and extended external sources can be found in the literature [8]-[33]. Likewise some exact retarded solutions to Yang-Mills equations with sources composed of N arbitrarily moving coloured point particles were studied in [34]. On the other hand some specific solutions of the $(2+1)$ -dimensional $SU(2)$ Yang-Mills equations were obtained in [35, 36]. The discovery of global regular solutions of $SU(2)$ Einstein-Yang-Mills equations [37] originated a great interest about spherical symmetric solutions [38]-[43]. For instance, the solutions of the reduced $SU(2)$ Einstein-Yang-Mills equations with spherical symmetry was presented in [44]. Additionally some spherical solutions were considered in [45] for the $SU(2)$ Einstein-Yang-Mills theory with a negative cosmological constant. Concurrently, it was given a rigorous proof for the existence of infinitely many black hole solutions to the $SU(3)$ Einstein-Yang-Mills equations [46]. Latterly, the study of quark confinement problem in curved space-time has been a subject of interest [47].

A semiclassical approach motivated by the black hole physics techniques was proposed by Goncharov [48, 49, 50] to describe the energy spectra of quarkonia by solving the Dirac equation in presence of $SU(3)$ Yang-Mills fields representing gluonic fields. In the context of this approach, explicit calculations have shown that gluon concentration is huge at scales of the order of 1 fm [51]. The solutions obtained there can model the quark confinement satisfactorily, suggesting that its mechanism might occur within the framework of QCD [52]. This implies that at large distances the gluons form a boson condensate and, therefore, gluons can be described by the classical $SU(3)$ -gauge fields [52]. For this reason the dynamics of the strong interaction at long distances could be described by equations of motion of the $SU(3)$ Yang-Mills theory [52].

In this work we follow the semiclassical approach proposed by Goncharov, presented in detail in [52], and we obtain some exact static solutions of the $(n + 1)$ -dimensional $SU(N)$ Yang-Mills equations in curved space-time, for the cases $n = 1, 2, 3$. As an application of the results obtained for the case $n = 3$, we consider the solutions for the anti-de Sitter and Schwarzschild metrics. We show that these solutions have a confining behavior and can be considered as a first step in the study of the corrections of the spectra of quarkonia in a curved background. Since the solutions that we find in this work are valid also for the group $U(1)$, the case $n = 2$ is a description of the $2 + 1$ electrodynamics in presence of a point charge. For this case, the solution has a confining behavior and can be considered as an

application of the planar electrodynamics in curved space-time. i.e. electrodynamics in two spatial dimensions (QED₂₊₁). For the case $n = 1$, we find that the solution is invariant under a parity transformation and have the form of a linear confining solutions.

The structure of this paper is as follows. In section 2 we first present some preliminary aspects related to basic definitions and notation. In section 3, we obtain the solutions for the $(n+1)$ -dimensional $SU(3)$ Yang-Mills equations in curved space-time for the cases $n = 1, 2, 3$. Finally our conclusions are summarized in section 4.

2. Preliminaries

We work on some spacetime manifold M where in local coordinates the line element looks as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

and the components $g_{\mu\nu}$ take different values depending on the choice of coordinates and dimensions. The *Hodge star operator* $*$ is defined by relation: $\Lambda^p(M) \rightarrow \Lambda^{n-p}(M)$, where $\Lambda^p(M)$ is the space of p -form over the manifold M under consideration. If $\{dx^1, \dots, dx^n\}$ is the base for $\Lambda^p(M)$ then

$$*(dx^{i_1} \wedge \dots \wedge dx^{i_p}) = \frac{g^{(1/2)}}{(n-p)!} g^{i_1 l_1} \dots g^{i_p l_p} \varepsilon_{l_1 \dots l_p l_{p+1} \dots l_n} dx^{l_{p+1}} \wedge \dots \wedge dx^{l_n}. \quad (2)$$

The *exterior differential*, which is represented by d , is defined as: $\Lambda^p(M) \rightarrow \Lambda^{p+1}(M)$. This means that

$$d = \partial_\mu dx^\mu. \quad (3)$$

If the connection A in the gauge group $SU(N)$ is defined as

$$A = A_\mu dx^\mu = A_\mu^a \lambda_a dx^\mu, \quad (4)$$

where A_μ^a are the non-abelian fields associated to $SU(N)$, λ_a are the generators of $SU(N)$, with $a = 1, 2, \dots, N^2 - 1$, then the curvature F can be defined using the exterior differential as

$$F = dA + gA \wedge A = F_{\mu\nu}^a \lambda_a dx^\mu \wedge dx^\nu, \quad (5)$$

where $F_{\mu\nu}^a$ represents the non-abelian stress tensor associated to the gauge group $SU(N)$.

It is possible to write the $SU(N)$ Yang-Mills equations of the non-abelian gauge field in presence of sources using the Hodge star operator as follows

$$d * F = g(*F \wedge A - A \wedge *F) + gJ, \quad (6)$$

where F is the curvature (5), A is the connection (4), g is the gauge coupling constant associated with the gauge group $SU(N)$ and, for example, if considering the QCD-lagrangian corresponding to that group then J is a nonabelian current given by

$$J = j_\mu^a \lambda_a * (dx^\mu) = *j = *(j_\mu^a \lambda_a dx^\mu) = \bar{\Psi}(I \otimes \gamma_\mu) \lambda^a \Psi \lambda_a dx^\mu, \quad (7)$$

where Ψ are the Dirac fields, γ^a are matrices which represent the Clifford algebra and I represents the identity. For the case of a point particle at rest, this current density is given by $J = *(j_\mu^a \lambda_a dx^\mu) = \delta(\vec{r}) q^a \lambda_a * dt$, where q^a are constants and then $q^a \lambda_a = \Upsilon$ is a constant. In what follows we put $J = \delta(\vec{r}) q^a \lambda_a * dt$.

In an analogous way as it is performed into the functional quantization procedure of Yang-Mills theories, we fix the gauge through the condition

$$\text{div}(A) = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} A_\nu) = 0. \quad (8)$$

3. Solutions in a $(n + 1)$ -dimensional curved space-time

The $SU(N)$ Yang-Mills equations are a non-linear system of coupled partial differential equations. In this section we will present some static and exact solutions of the $SU(N)$ Yang-Mills equations in a $(n + 1)$ dimensional curved space-time, for the cases $n = 1, 2, 3$.

3.1. Case $n = 3$

The metric for a curved static space-time and spherical symmetry is specified by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \alpha^2(r) dt^2 - \beta^2(r) dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (9)$$

Due to we are interesting in the description of problems which involve spherical symmetry in which the charge point is localized in $r = 0$, then we assume that the connection (4) has the following functional dependence

$$A = A_\mu(r) dx^\mu = A_t(r) dt + A_r(r) dr + A_\theta(r) d\theta + A_\varphi(r) d\varphi. \quad (10)$$

For this case the gauge condition (8) allow to the following equation

$$\partial_r \left(\frac{r^2 \alpha(r) A_r(r)}{\beta(r)} \right) + \alpha(r) \beta(r) A_\theta(r) \cot\theta = 0, \quad (11)$$

that can be satisfied if $A_\theta(r) = 0$ and $A_r(r) = C \frac{\beta(r)}{r^2 \alpha(r)}$. We can set $C = 0$ in the A_r solution because this fact does not affect the form of A_t and A_φ solutions. So we assume the following anzats for the form of these solutions [53]

$$A_t = f(r) \Gamma, \quad (12)$$

$$A_\varphi = g(r) \Delta, \quad (13)$$

where Γ and Δ are linear combinations of the group generators. As the exterior differential in spherical coordinates is

$$d = \partial_t dt + \partial_r dr + \partial_\theta d\theta + \partial_\varphi d\varphi, \quad (14)$$

then the curvature is given by

$$F = dA + gA \wedge A = -\partial_r f(r)\Gamma dt \wedge dr + \partial_r g(r)\Delta dr \wedge d\varphi + gf(r)g(r)[\Gamma, \Delta]dt \wedge d\varphi. \quad (15)$$

For this case the $SU(N)$ Yang-Mills are written by the equations (6), but with the connection given by (10) and the curvature by (15). Applying the Hodge star operator to F with inserting the result into (6) and using the following relations

$$\begin{aligned} *(dt \wedge dr) &= -\frac{r^2 \sin\theta}{\alpha(r)\beta(r)} d\theta \wedge d\varphi, \\ *(dt \wedge d\theta) &= \frac{\beta(r) \sin\theta}{\alpha(r)} dr \wedge d\varphi, \\ *(dt \wedge d\varphi) &= \frac{-\beta(r)}{\alpha(r) \sin\theta} dr \wedge d\theta, \\ *(dr \wedge d\varphi) &= \frac{-\alpha(r)}{\beta(r) \sin\theta} dt \wedge d\theta, \\ *(dr \wedge d\theta) &= \frac{\alpha(r) \sin\theta}{\beta(r)} dt \wedge d\varphi, \\ *(d\theta \wedge d\varphi) &= \frac{\alpha(r)\beta(r)}{r^2 \sin\theta} dt \wedge dr, \end{aligned}$$

we can obtain, for $r \neq 0$, that the $SU(N)$ Yang-Mills equations are reduced to the following coupled equation system

$$\partial_r \left(\frac{\alpha(r)\partial_r g(r)}{\beta(r)} \right) \Delta = -\beta(r) \frac{g^2}{\alpha(r)} f(r)^2 g(r) [\Gamma, [\Gamma, \Delta]], \quad (16)$$

$$\partial_r \left(\frac{r^2 \partial_r f(r)}{\alpha(r)\beta(r)} \sin^2 \theta \right) \Gamma = \frac{g^2 f(r) g(r)^2 \beta(r)}{\alpha(r)} [\Delta, [\Gamma, \Delta]] + g \frac{r^2 \beta(r) \delta(\vec{r})}{\alpha(r)} \Upsilon \sin^2 \theta. \quad (17)$$

A non-trivial solution from this equation can be obtained if the coupled equations (16) and (17) satisfy the abelian condition $[\Delta, [\Gamma, \Delta]] = 0$, then we obtain the two following independent equations

$$\partial_r \left(\frac{\alpha(r)\partial_r g(r)}{\beta(r)} \right) = 0, \quad (18)$$

$$\partial_r \left(\frac{r^2 \partial_r f(r)}{\alpha(r)\beta(r)} \right) = g \frac{r^2 \beta(0) \delta(\vec{r})}{\alpha(0)} \Upsilon, \quad (19)$$

that have the following solutions

$$g(r) = b_1 \int \frac{\beta(r)}{\alpha(r)} dr + B_1, \quad (20)$$

$$f(r) = a_1 \int \frac{\alpha(r)\beta(r)}{r^2} dr + A_1. \quad (21)$$

We note that the abelian condition $[\Gamma, \Delta] = 0$ is satisfied in a non-trivial way if and only if one of the following conditions hold:

1) If within the combinations of Γ and Δ , in terms of the generators of the group, it is present only the matrix that constitutes the Cartan subalgebra of the $SU(N)$ -Lie algebra i.e. a maximal abelian subalgebra. This means that the commutator of any two matrices of Cartan subalgebra is equal to zero.

2) If $\Gamma = k\Delta$ and k is a constant.

C.1. Case of flat space-time

We can obtain from (20) and (21) that the solutions, for the flat space-time case $\alpha = \beta = 1$, are

$$g(r) = b_1 r + B_1, \quad f(r) = -a_1/r + A_1. \quad (22)$$

These solutions are confining solutions in a semiclassical approach where the gluons can be considered as classical fields [48, 49, 50, 51, 52, 54, 55, 56, 57]. We have obtained the confining solutions (22) using the ansatz given by (12) and (13). We note that these solutions were obtained in [49], but using the ansatz

$$A = r^{\mu_a} \alpha^a \lambda_a dt + A_r dr + r^{\rho_a} \gamma^a \lambda_a d\theta + r^{\nu_a} \beta^a \lambda_a d\varphi, \quad (23)$$

where μ_a , ν_a , ρ_a , α^a , γ^a and β^a are arbitrary real constants. The confining potentials between quarks are usually modeled as $V(r) = A/r + Br$, however it has been shown that $A_t dt = (A/r + Br)dt$ is not a solution of the Yang-Mills equations in the presence of a point charge [54]. The solutions (22), that we have found here, were applied to describe the energy spectra of quarkonia (charmonium and bottomonium) in [50], to predict the electric form factor, the magnetic moment and the root-mean-square radius of mesons in [55, 56, 57], and to study the chiral symmetry breaking in QCD [58].

It was shown that the confining solutions (22) satisfy the so-called Wilson confinement criterion [52]. This criterion is in essence the assertion that the so-called Wilson loop $W(c)$ should be subjected to the area law for the confining gluonic field configuration. As a consequence, the latter law is equivalent to the fact that energy $E(R)$ of the mentioned configuration (gluon condensate) is linearly increasing with R , a characteristic size of some volume V containing the condensate [52]. The evaluation of $E(R)$ was carried out in [52]

using the T_{00} - component of the energy-momentum tensor for a $SU(3)$ -Yang-Mills field. However, formally $E(R)$ diverges for everything R . Calculations carried out in [52] have considered the integral about the angle θ inside the limits $(\theta_0, \pi - \theta_0)$, with the purpose of avoiding the divergence. Next we will consider a classical estimate of Wilson loop in the real-time formalism ($\tau \rightarrow it$) directly for the gluon condensate, i. e.

$$W(R, T) \equiv \langle e^{ig \oint A_\mu dx^\mu} \rangle = e^{-TV(R)}, \quad (24)$$

and taking into account that the expectation value coincides with the evaluation of the integral in this approach. We obtain as result that

$$\oint A_\mu dx^\mu = \int_0^{2\pi} (b_1 R + B_1) Tr \Delta d\varphi = 2\pi(b_1 R + B_1) Tr \Delta, \quad (25)$$

where Tr is the trace of the matrix. We observe from this result that the confinement potential obtained has the form $V(R) = \sigma R + C$. We note that this solution has the form of the confining Cornell potential [59] which has been used to describe experimental features of QCD. Lattice QCD simulations carried out have found the same kind of potential from Wilson loop [60, 61, 62].

C.2. Case of anti-de Sitter metric

As a particular application of the $(3+1)$ case solution, we will consider the anti-de Sitter metric given by

$$ds^2 = (1 - \Lambda r^2/3)dt^2 - (1 - \Lambda r^2/3)^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (26)$$

As $\alpha(r)$ is the inverse of $\beta(r)$, then the Coulomb solution has not deformations respect to the flat space-time case, then the solution of the function $f(r)$ is given by $f(r) = a_1/r + A_1$. On the other hand, the linear solution $g(r)$ changes respect to the flat case. We get the solution explicitly in the two following situations $\Lambda > 0$ and $\Lambda < 0$, so

$$g(r) = \begin{cases} b_1 \tanh^{-1} \left(\frac{r\Lambda^{1/2}}{3^{1/2}} \right) + B_1, & \text{if } \Lambda > 0, \\ b_1 \tan^{-1} \left(\frac{r(-\Lambda)^{1/2}}{3^{1/2}} \right) + B_1, & \text{if } \Lambda < 0, \end{cases} \quad (27)$$

The function $g(r)$, for the limit cases $|\Lambda| \ll 1$ and $r \ll 1$, has the form $g(r) \simeq b_1 r + B_1$, recovering the flat space-time case behavior.

C.3. Case of Schwarzschild metric

Another application for the $(3+1)$ curved space-time solution is the Schwarzschild metric given by

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (28)$$

As in the last application, the Coulomb solution has not deformations respect to the flat space-time case, but the linear solution has. The function $g(r)$ for this case is

$$g(r) = b_1(r + 2M \ln |r - 2M|) + B_1. \quad (29)$$

For the limit $r \gg M$, this solution has the form $g(r) \simeq b_1 r + B_1$, recovering the flat space-time case behavior.

We observe that the solutions (27) and (29) are confining solutions. From the analysis of the spectra of quarkonia in a spherical symmetry background, we can observe that the linear solution changes as a consequence of the curvature. This fact could generate a change in the mass spectrum of mesons. This change additionally could be generated by the effects of the curvature in the Dirac equation. By this reason, the solutions (20) and (21) can represent a first step in the study of hadronic spectrum in a space-time curved.

3.2. Case $n = 2$

We consider now a curved space-time in $(2 + 1)$ dimensions defined by the metrics

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \alpha^2(r) dt^2 - \beta^2(r) dr^2 - r^2 d\theta^2. \quad (30)$$

For this case we assume that the connection (4) has the following form

$$A = A_\mu(r) dx^\mu = A_t(r) dt + A_r(r) dr + A_\theta(r, \theta) d\theta. \quad (31)$$

Using the gauge condition (8), we can obtain the following equation

$$\partial_r \left(\frac{r\alpha(r)A_r(r)}{\beta(r)} \right) + \partial_\theta \left(\frac{\alpha(r)\beta(r)A_\theta(r, \theta)}{r} \right) = 0. \quad (32)$$

In the last equation, the solutions of the form $A_\theta(r, \theta) = H(r)\theta + G(r)$ can be discarded because they must be periodic in θ , i. e. A_θ does not depend on θ . Thus, the equation (32) can be written as

$$\partial_r \left(\frac{r\alpha(r)A_r(r)}{\beta(r)} \right) = 0. \quad (33)$$

The solution of this equation has the form $A_r = C \frac{\beta(r)}{r\alpha(r)}$. In these coordinates, the exterior differential is written as

$$d = \partial_t dt + \partial_r dr + \partial_\theta d\theta. \quad (34)$$

If now we set $C = 0$ and we substitute the functions $A_t = f(r)\Gamma$ and $A_\theta = g(r)\Delta$ in (31), where Γ and Δ are linear combinations of the group generators, and additionally we consider

the following relations

$$\begin{aligned} *(dt \wedge dr) &= -\frac{r}{\alpha(r)\beta(r)}d\theta, \\ *(dt \wedge d\theta) &= \frac{\alpha(r)\beta(r)}{r}dr, \\ *(dr \wedge d\theta) &= \frac{\alpha(r)}{r\beta(r)}dt, \end{aligned}$$

then the curvature is given by

$$F = dA + gA \wedge A = -\partial_r f(r)\Gamma dt \wedge dr + \partial_r g(r)\Delta dr \wedge d\varphi + gf(r)g(r)[\Gamma, \Delta]dt \wedge d\theta, \quad (35)$$

and the Hodge star operator applied over the curvature can be written as

$$*F = \frac{r\partial_r f(r)\Gamma}{\alpha(r)\beta(r)}d\theta + \frac{\alpha(r)}{r\beta(r)}\frac{\partial_r g(r)}{r}\Delta dt + g\frac{\alpha(r)\beta(r)f(r)g(r)}{r}[\Gamma, \Delta]dr. \quad (36)$$

The Yang-Mills equation for this case can be obtained from the expression (6) and has the form

$$\begin{aligned} &\partial_r \left(r \frac{\partial_r f(r)}{\alpha(r)\beta(r)} \Gamma \right) dr \wedge d\theta - \partial_r \left(\frac{\alpha(r)\partial_r g(r)}{r\beta(r)} \Delta \right) dt \wedge dr = g\delta(\vec{r}) \frac{r\beta(r)}{\alpha(r)} \Upsilon dr \wedge d\theta \\ &+ g^2 \left(\frac{f(r)g(r)^2\alpha(r)\beta(r)}{r} [[\Gamma, \Delta], \Delta] dr \wedge d\theta - \frac{f(r)^2g(r)\alpha(r)\beta(r)}{r} [[\Gamma, \Delta], \Gamma] dt \wedge dr \right). \end{aligned} \quad (37)$$

If the abelian condition is satisfied in the last equation, i. e. the commutator $[\Gamma, \Delta] = 0$, then the solutions will be strongly restricted by this condition. Thus, we can obtain the two following independent equations

$$\partial_r \left(\frac{\alpha(r)}{r\beta(r)} \partial_r g(r) \right) = 0, \quad (38)$$

$$\partial_r \left(\frac{r}{\alpha(r)\beta(r)} \partial_r f(r) \right) = g\delta(\vec{r}) \frac{r\beta(0)}{\alpha(0)} \Upsilon, \quad (39)$$

which solutions are given by

$$g(r) = k_2 \int \frac{r\beta(r)}{\alpha(r)} dr + d_2, \quad (40)$$

$$f(r) = k_1 \int \frac{\alpha(r)\beta(r)}{r} dr + d_1. \quad (41)$$

For $\alpha = \beta = 1$, these solutions can be written as

$$g(r) = d_2 + k_2 r^2, \quad (42)$$

$$f(r) = d_1 + k_1 \log r. \quad (43)$$

We observe that the function $f(r)$ has the form of the well known confining solution for the two dimensional problem [63]. Similarly, the function $g(r)$ has a form of a confining function. It is possible to eliminate the constant d_1 by means of a gauge transformation, so the potential is given by

$$A_\mu dx^\mu = k_1 \Gamma \log r dt + (d_2 + k_2 r^2) \Delta d\varphi, \quad (44)$$

and the 1-form vectorial field is

$$A_\mu = \left\{ k_1 \Gamma \log r, 0, \left(\frac{d_2}{r} + k_2 r \right) \Delta \right\}. \quad (45)$$

In the last expression, the term $k_1 \Gamma \log r$ is a potential equivalent to the Coulomb potential in $(2+1)$ dimensions and additionally it corresponds to the potential which is obtained for an infinite line of charge in $(3+1)$ dimensions. Additionally, the term $k_2 r$ in the z axis is equivalent to a constant magnetic field of magnitude $B_1 = 2k_2$ in $(3+1)$ dimensions. We can see that if we perform a transformation to cartesian coordinates over this last term, we obtain its usual representation

$$k_2 r \hat{e}_\varphi = \frac{B_1}{2} r (-\sin \varphi \hat{e}_x + \cos \varphi \hat{e}_y) = -\frac{B_1}{2} y \hat{e}_x + \frac{B_1}{2} x \hat{e}_y. \quad (46)$$

Finally, the term $\frac{d_2}{r}$ can be interpreted as the potential produced by an infinite solenoid of radius R in $(3+1)$ dimensions, with $R \rightarrow 0$. This last term can be written in its usual representation by means of a transformation to cartesian coordinates

$$\frac{d_2}{r} \hat{e}_\varphi = \frac{d_2}{r} (-\sin \varphi \hat{e}_x + \cos \varphi \hat{e}_y) = -\frac{d_2}{r^2} y \hat{e}_x + \frac{d_2}{r^2} x \hat{e}_y. \quad (47)$$

This result is known as the Aharonov-Bohm (AB) potential, where the magnetic field within the solenoid is $B_2 = d_2$.

Let's consider now the integral that appears in the Wilson loop

$$\oint A_\mu dx^\mu. \quad (48)$$

This integral can be performed in a circular path of radius R which is centered in the origin of the coordinates frame. Then we obtain that

$$\oint A_\mu dx^\mu = \int_0^{2\pi} (d_2 + k_2 R^2) Tr \Delta d\varphi = 2\pi (d_2 + k_2 R^2) Tr \Delta. \quad (49)$$

This result implies that for this approach the confinement potential in low dimensionality is given by $V(R) = \sigma R^2 + B$.

The result obtained for this case correspond to a $U(1)$ abelian solution which has a confining behavior and can be considered as an application of the planar electrodynamics, i.e. electrodynamics in two spatial dimensions (QED_{2+1}). QED_{2+1} has been worked in many emblem quantum systems such as the quantum Hall effect, the theory of anyons and the relativistic quantum Hall effect [64, 65, 66]. During the last years the physics of graphene has attracted considerable interest, both theoretically and experimentally [67, 68]. Due to the low energy excitations of graphene can be described by a massless Dirac equation in two spacial dimensions, the curved graphene has been modeled by coupling the Dirac equation to the corresponding curved space [69]. In the approach presented in [70], gauge fields has been considered as external fields into the Dirac equation and it was possible to model some topological defects in the graphene. In connection with the results that we present in this work, the solutions that we obtain for the case $n = 2$ can be interpreted as a gauge field which is affected by the curvature of graphene.

3.3. Case $n = 1$

For this case we start from the metric given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \alpha^2(x)dt^2 - \beta^2(x)dx^2. \quad (50)$$

We suppose that the connection has the following functional dependence

$$A = A_\mu(x)dx^\mu = A_t(x)dt + A_x(x)dx = \lambda_a f^a(x)dt + A_x(x)dx, \quad (51)$$

where λ_a are the group generators. Each of these generators has associated a function $f^a(x)$. The gauge condition (8) implies that

$$\partial_x \left(\frac{\alpha(x)A_x(x)}{\beta(x)} \right) = 0, \quad (52)$$

and then $A_x = C\beta(x)/\alpha(x)$. On this coordinates, the exterior differential is written as

$$d = \partial_t dt + \partial_x dx. \quad (53)$$

Fixing $C = 0$ and using the fact that $*(dt \wedge dx) = -\frac{1}{\alpha(x)\beta(x)}$, it is possible to write

$$*F = \frac{\lambda_a \partial_x f^a(x)}{\alpha(x)\beta(x)}. \quad (54)$$

For this case we have that $*(dt) = \frac{\beta(x)}{\alpha(x)}dx$ and the expression (6) leads us to

$$\partial_x \left(\frac{\lambda_a \partial_x f^a(x)}{\alpha(x)\beta(x)} \right) dx = g \left(\frac{\partial_x f^a(x)}{\alpha(x)\beta(x)} f^b(x) [\lambda_a, \lambda_b] \right) dt + g\delta(x) \frac{\beta(x)}{\alpha(x)} q^a \lambda_a dx. \quad (55)$$

A non-trivial solution from this equation can be obtained if the condition $[\lambda_a, \lambda_b] = 0$ is satisfied, or alternatively $\lambda_a \partial_x f^a(x) = \partial_x f(x) \Gamma$, where Γ represents a linear combination of the group generators. Thus, the last equation can be written as

$$\partial_x \left(\frac{\lambda_a \partial_x f^a(x)}{\alpha(x)\beta(x)} \right) = -g\delta(x) \frac{\beta(0)}{\alpha(0)} q^a \lambda_a. \quad (56)$$

We solve this equation and obtain

$$f^a(x) = k^a \int \alpha(x)\beta(x)dx + d^a, \quad (57)$$

where k^a and d^a are constants. For $\alpha = \beta = 1$ we can write $f(x) = d + k|x|$, i. e. we have obtained the solution of the flat space-time case. This last solution is invariant under a parity transformation and has the form of a linear confining solution. The form of this solution corresponds to the potential that results due to an infinite sheet of charge in $(3+1)$ dimensions.

4. Conclusions

We have obtained some exact static solutions for the $SU(N)$ Yang-Mills equations in a curved space-time of $(n+1)$ dimensions for the cases $n = 1, 2, 3$. For the $(1+1)$ case, we have found that the solution for the temporary part can be written as $A_t = f(r)\Gamma$. This solution is the most general static solution in a space-time in presence of a point charge. To be able to find analytic solutions in the $(2+1)$ case, it was necessary to demand that the abelian condition given by $[\Delta, \Gamma] = 0$ were satisfied into the Yang-Mills equation for this case. For the cases $(1+1)$ and $(3+1)$, the abelian condition was naturally satisfied. We have presented in detail the solution for $(3+1)$ curved space-time case and we have applied this solution to the anti-de Sitter and Schwarzschild cases. In both cases, the Coulomb solutions have not deformations respect to the flat space-time case, while in the linear solution exists deformation. We have assumed that these solutions can be considered as a first step in the study of the corrections on the spectra of quarkonia in a curved background. Although the energy diverges for the solutions that we have found, this fact is not a problem since this energy behavior was already known and it is not an impediment to make physical predictions in agreement with experimental data [50, 54, 55, 56, 57, 58]. Since the solutions that we have

found here are valid also for the group $U(1)$, the case $n = 2$ is a description of the $(2 + 1)$ electrodynamics in presence of a point charge. Finally, we have found that the solution for the case $n = 1$ is invariant under a parity transformation and has the form of a linear confining solution. As a perspective of this work, it would be interesting to understand the role of the confining solutions in a model of relativistic quark confinement in low dimensionality and the spectra of quarkonia in a curved background.

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